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Abstract

Graph eigenvalues are examples of totally real algebraic integers, i.e. roots of real-rooted monic polynomials with integer coefficients. Conversely, the fact that every totally real algebraic integer occurs as an eigenvalue of some finite graph is a deep result, conjectured forty years ago by Hoffman, and proved seventeen years later by Estes. This short paper provides an independent and elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree. As a by-product, our result implies that the atoms of the limiting spectrum of $n \times n$ symmetric matrices with independent Bernoulli $\left(\frac{c}{n}\right)$ entries ($c > 0$ is fixed as $n \to \infty$) are exactly the totally real algebraic integers. This settles an open problem raised by Ben Arous (2010).

Keywords: adjacency matrix, tree, eigenvalue, totally real algebraic integer
2010 MSC: 05C05, 05C25, 05C31, 05C50

1. Introduction

By definition, the *eigenvalues* of a finite graph $G = (V, E)$ are the roots of its characteristic polynomial $\Phi_G(x) := \det(xI - A)$, where $A = \{A_{i,j}\}_{i,j \in V}$ is the adjacency matrix of G:

$$A_{i,j} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Those eigenvalues capture a considerable amount of information about G. For a detailed account, see e.g. [1, 2]. It follows directly from this definition that any graph eigenvalue is a *totally real algebraic integer*, i.e. a root of some real-rooted monic polynomial with integer coefficients. Remarkably enough,

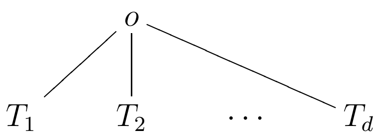
URL: `http://www.proba.jussieu.fr/~salez/` (Justin Salez) the converse is also true: every totally real algebraic integer is an eigenvalue of some finite graph. This deep result was conjectured forty years ago by Hoffman [3], and established seventeen years later by Estes [4], see also [5]. The present paper provides an independent, elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree.

Theorem 1. *Every totally real algebraic integer is an eigenvalue of some finite tree.*

Trees undoubtedly play a special role in many aspects of graph theory. We therefore believe that the strengthening provided by Theorem 1 may be of independent interest, beyond the fact that it provides a simpler proof of Hoffman's conjecture. In addition, Theorem 1 settles an open problem raised by Ben Arous [6, Problem 14], namely that of determining the set Σ of atoms of the limiting spectral distribution of $n \times n$ symmetric matrices with independent Bernoulli $\left(\frac{c}{n}\right)$ entries ($c > 0$ is fixed and $n \to \infty$). Indeed, it follows from the log-Hölder continuity of the spectrum of integer matrices at algebraic numbers (see e.g. [7, Section 6]) that Σ is contained in the set of totally real algebraic integers. On the other hand, Σ is easily seen to contain all tree eigenvalues, as noted by Ben Arous. Theorem 1 precisely states that those inner and outer bounds coincide, thereby settling the question.

2. Outline of the proof

Let T be a finite tree with a distinguished vertex o (the root). Removing o naturally yields a decomposition of T into smaller rooted trees $T_1, \ldots, T_d$ ($d \in \mathbb{N}$) as depicted in the following diagram:



To such a rooted tree, let us associate the rational function

$$\mathfrak{f}_T(x) \;=\; 1 - \frac{\Phi_T(x)}{x\Phi_{T\setminus o}(x)} \;=\; 1 - \frac{\Phi_T(x)}{x\Phi_{T_1}(x)\cdots\Phi_{T_d}(x)}. \tag{1}$$

Expressed in terms of this function, the classical recursion for the characteristic polynomial of trees (see e.g. [2, Proposition 5.1.1]) simply reads

$$\mathfrak{f}_T(x) \;=\; \frac{1}{x^2}\sum_{i=1}^{d}\frac{1}{1-\mathfrak{f}_{T_i}(x)}, \tag{2}$$

the sum being interpreted as 0 when empty (i.e. when T is reduced to o). Now fix $\lambda \in \mathbb{C} \setminus \{0\}$. In view of (1), the problem of finding a (minimal) tree with eigenvalue λ is equivalent to that of finding T such that $\mathfrak{f}_T(\lambda) = 1$. Thanks to the recursion (2), this task boils down to that of generating the number 1 from the initial seed 0 using repeated applications of the maps

$$(\alpha_1, \ldots, \alpha_d) \quad \longmapsto \quad \frac{1}{\lambda^2} \sum_{i=1}^{d} \frac{1}{1 - \alpha_i} \qquad (d \in \mathbb{N}).$$

As an example, consider the golden ratio $\lambda = \frac{1+\sqrt{5}}{2}$. Iterating 4 times the map $\alpha \mapsto \frac{1}{\lambda^2(1-\alpha)}$ and using the identity $\lambda^2 = \lambda + 1$ yields successively:

$$0 \;\longrightarrow\; \tfrac{1}{\lambda^2} \;\longrightarrow\; \tfrac{1}{\lambda^2 - 1} = \tfrac{1}{\lambda} \;\longrightarrow\; \tfrac{1}{\lambda^2 - \lambda} = 1,$$

which shows that λ is an eigenvalue of the linear tree $T = P_4$. Remarkably enough, this seemingly specific argument can be extended in a systematic way to any totally real algebraic integer λ, and the numbers that may be produced in this way can be completely determined. To formalize this, let us fix a totally real algebraic integer $\lambda \neq 0$, and introduce the rational function

$$\Psi(x) \quad = \quad \frac{1}{\lambda^2(1 - x)}. \tag{3}$$

Define $\mathcal{F} \subseteq \mathbb{R}$ as the smallest set containing 0 and satisfying for all $d \geq 1$,

$$\alpha_1, \ldots, \alpha_d \in \mathcal{F} \setminus \{1\} \implies \Psi(\alpha_1) + \cdots + \Psi(\alpha_d) \in \mathcal{F}. \tag{4}$$

We will prove that $\mathcal{F}$ is nothing but the field generated by λ^2:

$$\mathcal{F} = \left\{ \frac{P(\lambda^2)}{Q(\lambda^2)} \colon P, Q \in \mathbb{Z}[x], Q(\lambda^2) \neq 0 \right\} =: \mathbb{Q}(\lambda^2). \tag{5}$$

In particular, $1 \in \mathcal{F}$, and Theorem 1 follows. The remainder of the paper is devoted to the proof of (5). The detailed argument is given in Section 4, while Section 3 provides the necessary background on algebraic numbers.

3. Algebraic preliminaries

A number $\zeta \in \mathbb{C}$ is *algebraic* if there is $P \in \mathbb{Q}[x]$ such that $P(\zeta) = 0$. In that case, such P are exactly the multiples of a unique monic polynomial $P_0 \in \mathbb{Q}[x]$, called the *minimal polynomial* of ζ. The algebraic number ζ is

- *totally real* if all the complex roots of P_0 are real ;
- *totally positive* if all the complex roots of P_0 are real and positive.

The following Lemma gathers the basic properties of algebraic numbers that will be used in the sequel. These are well-known (see e.g. [8]) and follow directly from the fact that if $P(x) = \prod_i (x-\alpha_i)$ and $Q(x) = \prod_j (x-\beta_j)$ have rational coefficients, then so do the polynomials

$$\prod_{i,j}(x-\alpha_i-\beta_j), \qquad \prod_i \left(x-\frac{1}{\alpha_i}\right), \qquad \prod_{i,j}(x-\alpha_i\beta_j)$$
$$\prod_i (x+\alpha_i), \qquad \prod_i (x^2-\alpha_i^2), \qquad \prod_i (x^2-\alpha_i).$$

Lemma 1 (Elementary algebraic properties)**.**

a. *The totally real algebraic numbers form a sub-field of* $\mathbb{R}$.
b. *The set of totally positive algebraic numbers is stable under* $+, \times, \div$.
c. *If α is totally real and β is totally positive, then $\alpha+n\beta$ is totally positive for all sufficiently large $n \in \mathbb{N}$.*
d. *If $\alpha \neq 0$ is totally real, then α^2 is totally positive.*

We shall also use twice the following elementary result.

Lemma 2. *Let ζ be algebraic with minimal polynomial P. Set $n = \deg P$. Then for any $q_1 > \ldots > q_n \in \mathbb{Q}$, there exist $m_1, \ldots, m_n \in \mathbb{Z}$ such that*

$$\frac{m_1}{\zeta - q_1} + \cdots + \frac{m_n}{\zeta - q_n} \quad \in \quad \mathbb{N}^+ = \{1, 2, \ldots\}, \tag{6}$$

and m_k has the same sign as $(-1)^k P(q_k)$ for every $k \in \{1, \ldots, n\}$.

Proof. For $1 \leq k \leq n$, consider the rational number

$$r_k \quad := \quad \frac{-P(q_k)}{\prod_{j \neq k}(q_k - q_j)}.$$

Note that r_k has the same sign as $(-1)^k P(q_k)$, since $q_1 > \ldots > q_n$. Moreover,

$$r_1 \prod_{k \neq 1}(x-q_k) + \cdots + r_n \prod_{k \neq n}(x - q_k) \quad = \quad \prod_{k=1}^{n}(x-q_k) - P(x).$$

Indeed, both sides are polynomials of degree less than n, and they coincide at the n points $q_1, \ldots, q_n$. Evaluating at $x = \zeta$ gives

$$\frac{r_1}{\zeta - q_1} + \cdots + \frac{r_n}{\zeta - q_n} \quad = \quad 1,$$

and multiplying this by a large enough integer yields the result. □

4. Proof

Before we start, let us make three simple observations which will be used several times in the sequel. First, by (4), we have for any $k \in \mathbb{N}$,

$$\frac{1}{\lambda^2 - k} \quad = \quad \Psi\left(\underbrace{\Psi(0) + \cdots + \Psi(0)}_{\text{k terms}}\right) \in \mathcal{F}. \tag{7}$$

Second, $\mathcal{F}$ is stable under internal addition:

$$\alpha, \beta \in \mathcal{F} \implies \alpha + \beta \in \mathcal{F}. \tag{8}$$

Indeed, the conclusion is trivial if $\alpha = 0$ or $\beta = 0$. Now if α, β are non-zero elements of $\mathcal{F}$, then by construction they are of the form

$$\alpha = \Psi(\alpha_1) + \cdots + \Psi(\alpha_n) \quad \text{and} \quad \beta = \Psi(\beta_1) + \cdots + \Psi(\beta_m),$$

for some $n, m \geq 1$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ in $\mathcal{F} \setminus \{1\}$. But then,

$$\alpha + \beta \quad = \quad \Psi(\alpha_1) + \cdots + \Psi(\alpha_n) + \Psi(\beta_1) + \cdots + \Psi(\beta_m),$$

which, by (4), shows that $\alpha + \beta \in \mathcal{F}$. Third, the field $\mathbb{Q}(\lambda^2)$ obviously contains 0 and satisfies (4). Since $\mathcal{F}$ is minimal with this property, we deduce that

$$\mathcal{F} \subseteq \mathbb{Q}(\lambda^2). \tag{9}$$

By part (a) of Lemma 2, we also get that all elements in $\mathcal{F}$ are totally real.

Step 1: $\mathcal{F}$ contains a positive integer

We may assume that $1 \notin \mathcal{F}$, otherwise there is nothing to prove. Consequently, we do not need to worry about divisions by zero when applying Ψ to an element $\alpha \in \mathcal{F}$. Let us first apply Lemma 2 to $\zeta = \lambda^2$ with $q_k = k, 1 \le k \le \deg \zeta$. From (7) and (8), it follows that the sum appearing in (6) is the difference of two elements in $\mathcal{F}$. In other words, we have found $\Delta \in \mathbb{N}^+$ and α with the following property:

$$\alpha \in \mathcal{F} \quad \text{and} \quad \alpha - \Delta \in \mathcal{F}. \tag{10}$$

As already noted, α is totally real. In fact we may even assume that $1 - \alpha$ is totally positive, because $\alpha' = \alpha + \beta$ also satisfies (10) for any $\beta \in \mathcal{F}$, and choosing $\beta = \frac{j}{\lambda^2 - k}$ with $j, k \in \mathbb{N}$ large enough eventually makes $1 - \alpha'$ totally positive, by parts (b) and (c) of Lemma 1. In turn, parts (b) and (d) now guarantee that

$$\xi := \lambda^2(1 - \alpha) \quad \text{is totally positive.} \tag{11}$$

Now fix $j, k \in \mathbb{N}$ and set $i = (\Delta - 1)j + 1$. Since $\mathcal{F}$ is stable under addition, property (10) and the fact that $\frac{1}{\lambda^2} = \Psi(0) \in \mathcal{F}$ imply that

$$\underbrace{\alpha + \cdots + \alpha}_{i \text{ terms}} + \underbrace{(\alpha - \Delta) + \cdots + (\alpha - \Delta)}_{j \text{ terms}} + \underbrace{\frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^2}}_{k \text{ terms}} \ \in \ \mathcal{F}.$$

But this number equals $1 - (\Delta j + 1)(1 - \alpha) + \frac{k}{\lambda^2}$, so applying Ψ gives

$$\frac{1}{(\Delta j + 1)\xi - k} \ \in \ \mathcal{F}.$$

Adding up $\Delta j + 1$ copies of this last number, we finally arrive at

$$\frac{1}{\xi - q} \in \mathcal{F} \quad \text{for any} \quad q \in \mathcal{Q} := \left\{ \frac{k}{\Delta j + 1} : j, k \in \mathbb{N} \right\}. \tag{12}$$

We may now conclude: by (11), the minimal polynomial P of ξ has $n := \deg P$ pairwise distinct positive roots. Since $\mathcal{Q}$ is dense in $[0, \infty)$, one can find $q_1 > \cdots > q_n$ in $\mathcal{Q}$ that interleave those roots, in the sense that $P(q_k)$ has sign $(-1)^k$ for every $1 \le k \le n$. Consequently, Lemma 2 provides us with non-negative integers $m_1, \ldots, m_n$ such that

$$\frac{m_1}{\xi - q_1} + \cdots + \frac{m_n}{\xi - q_n} \in \mathbb{N}^+.$$

On the other hand, this sum is in $\mathcal{F}$ by (12) and (8). Thus, $\mathcal{F} \cap \mathbb{N}^+ \neq \emptyset$.

Step 2: $(\mathcal{F}, +)$ is a group

We know that $\mathcal{F}$ contains some $d \in \mathbb{N}^+$. Since $d + d$ also belongs to $\mathcal{F}$, we may assume without loss of generality that $d \neq 1$ to avoid divisions by zero below. Now fix $\alpha \in -\mathcal{F}$ with $\alpha \neq 1$. Since $\mathcal{F}$ is stable under addition,

$$d + \underbrace{(-\alpha) + \cdots + (-\alpha)}_{d-1 \text{ terms}} \in \mathcal{F}.$$

Applying Ψ shows that $\frac{-1}{\lambda^2(1-\alpha)(d-1)} \in \mathcal{F}$. Adding up $(d-1)$ copies of this number, we conclude that $\frac{-1}{\lambda^2(1-\alpha)} \in \mathcal{F}$. We have proved:

$$\alpha \in (-\mathcal{F}) \setminus \{1\} \implies \Psi(\alpha) \in -\mathcal{F}.$$

In other words, $-\mathcal{F}$ is stable under Ψ. In view of (8), we deduce that $-\mathcal{F}$ satisfies (4). By minimality of $\mathcal{F}$, we conclude that $\mathcal{F} \subseteq -\mathcal{F}$, i.e. that $\mathcal{F}$ is stable under negation. Thus, the monoid $(\mathcal{F}, +)$ is a group.

Step 3: $\mathcal{F}$ is the field $\mathbb{Q}(\lambda^2)$.

In view of (9), we only need to show that for $P, Q \in \mathbb{Z}[x]$ with $Q(\lambda^2) \neq 0$,

$$\frac{P(\lambda^2)}{Q(\lambda^2)} \in \mathcal{F}. \tag{13}$$

Since λ^2 is an algebraic integer, we may assume that Q is monic with $\deg Q > \deg P$ (otherwise, replace Q with $Q + P_0^{\deg P}$, where P_0 denotes the minimal polynomial of λ^2). Let us prove the claim by induction over $n = \deg Q$. The case $n = 0$ is simply the fact that $0 \in \mathcal{F}$. Now, assume that the claim holds for a certain $n \in \mathbb{N}$, and consider

$$Q(x) = x^{n+1} + a_n x^n + \cdots + a_0,$$

with $a_0, \ldots, a_n \in \mathbb{Z}$. Let us first prove (13) in the following two special cases:

- Case 1: $P(x) = x^n$. By our induction hypothesis, $\frac{1}{1+\lambda^{2n}} \in \mathcal{F}$ and hence

$$\frac{1}{\lambda^{2n+2}} + \frac{1}{\lambda^2} = \Psi\left(\frac{1}{1+\lambda^{2n}}\right) \in \mathcal{F}.$$

But $\mathcal{F}$ also contains $\frac{1}{\lambda^2}, \ldots, \frac{1}{\lambda^{2n}}$ by our induction hypothesis. Since $(\mathcal{F}, +)$ is a group, one deduces that $\mathcal{F}$ contains

$$-\left(\frac{a_n}{\lambda^2} + \cdots + \frac{a_0}{\lambda^{2n+2}}\right) = 1 - \frac{Q(\lambda^2)}{\lambda^{2n+2}}.$$

Finally, applying Ψ shows that $\frac{\lambda^{2n}}{Q(\lambda^2)} = \frac{P(\lambda^2)}{Q(\lambda^2)} \in \mathcal{F}$, as desired.

- Case 2: P is monic of degree n with $P(0) = 1$. Then

$$R(x) := P(x) - \frac{Q(x) - Q(0)P(x)}{x}$$

is a polynomial over $\mathbb{Z}$ with $\deg R < n$. Thus, our induction hypothesis guarantees that $\mathcal{F}$ contains $\frac{R(\lambda^2)}{P(\lambda^2)}$, hence $\frac{R(\lambda^2)}{P(\lambda^2)} - \frac{Q(0)}{\lambda^2}$ and hence also

$$\Psi\left(\frac{R(\lambda^2)}{P(\lambda^2)} - \frac{Q(0)}{\lambda^2}\right) = \frac{P(\lambda^2)}{Q(\lambda^2)}.$$

For the general case, note that every monomial x^k $(0 \leq k \leq n)$ may be written as a signed sum of polynomials P of the form considered in the two special cases above. Since $(\mathcal{F}, +)$ is a group, we conclude that (13) holds for all $P \in \mathbb{Z}[x]$ with $\deg P \leq n$, and the induction is complete.

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